

Continuous non-perturbative regularization of QED

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Abstract

We regularize in a continuous manner the path integral of QED by construction of a non-local version of its action by means of a regularized form of Dirac's δ functions. Since the action and the measure are both invariant under the gauge group, this regularization scheme is intrinsically non-perturbative. Despite the fact that the non-local action converges formally to the local one as the cutoff goes to infinity, the regularized theory keeps trace of the non-locality through the appearance of a quadratic divergence in the transverse part of the polarization operator. This term which is uniquely defined by the choice of the cutoff functions can be removed by a redefinition of the regularized action. We notice that as for chiral fermions on the lattice, there is an obstruction to construct a continuous and non ambiguous regularization in four dimensions. With the help of the regularized equations of motion, we calculate the one particle irreducible functions which are known to be divergent by naive power counting at the one loop order.

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1 Introduction

As it is well known gauge invariant quantum field theory is a powerful tool to describe fundamental interactions in physics. However these theories suffer from ultraviolet (UV) divergences if we impose the locality of the interactions. Over the past decade since the birth of QED great efforts have been made in order to regularize these theories in a gauge invariant manner in order to preserve the Ward-Takahashi (WT) identities, since it appears hopeless to cure this problem by the construction of non local interactions [1]. This is a necessity dictated by the renormalization program. There are essentially two categories of gauge invariant regularization schemes, a discrete one and a continuous one. In the first category the path integral of the theory is discretized on a lattice. This allows to make non-perturbative, but mostly non analytical predictions. On the other hand the second category of continuous gauge invariant regularization schemes is intrinsically perturbative, in the sense that in momentum space these schemes basically regularize the Feynman integrals or the lowest order Green's functions in configuration space. This is the case for Pauli-Villars regularization [2], dimensional regularization [3, 4], Schwinger's proper time methods inspired regularization [5], ζ function regularization [6], differential regularization [7] and Fujikawa's type of regularization [8, 9].

If one wants to study the non-perturbative properties or the different phases of a gauge theory as a function of the scale, a general method is the Wilson-Kadanoff [10] approach. In this case it is more sensible to work, in four dimensions, with a continuous regularized and gauge invariant version of the path integral of the theory. In recent years, some progress have been made in this direction. For instance, gauge invariant regularization of chiral gauge theories can be achieved by a generalization of the Pauli-Villars method [11, 9] i.e. by the addition of a finite or infinite set of non physical regulator fields at the lagrangian level. In the same spirit a generalization of Schwinger's proper time methods, the operator cutoff regularization [12], allows to calculate effective gauge invariant regularized action at the one loop level.

In the present work we follow Wilson's idea, but we regularize non-perturbatively the path integral of QED in a continuous manner. This is done by a suited smearing of the point-like interactions of the fields by cutoff functions which are a regularized form of Dirac's δ function. We choose the cutoff functions to act only in the (UV) domain and not in the infrared (IR) one. In this sense, for finite (UV) cutoff our action is non-local, but converges asymptotically to the local one as the cutoff goes to infinity. We should notice that an earlier attempt was made in constructing a non-local regularization of gauge invariant theory [13]. In this regularization scheme only the action is invariant under a non-local transformation¹

¹This transformation is not the representation of a group.

which plays the role of the gauge transformation. The invariance of the path integral is maintained by adding a term in the action which cancels the contribution of the transformed measure. Since this term must be worked out order by order in perturbation theory, this method belongs to the category of perturbative regularization schemes.

In our case, both the cutoff action and the measure are invariant under the gauge group, therefore there is no need to define the regularized partition function perturbatively through the regularization of the Feynman diagrams. As a result the path integral is regularized as a whole in a gauge invariant manner. This feature gives us the opportunity to apply directly non-perturbative techniques to the regularized partition function in order to derive the physical properties of the theory. For instance, in addition to numerical simulations, we can study the variation of the path integral under an infinitesimal variation of the cutoff scale in order to derive Polchinski's [14] like exact renormalization group equation. We can also derive the regularized equations of motion, i.e. the regularized Dyson-Schwinger (DS) equations, directly from the regularized partition function of the theory. Since this scheme keeps track of the dimension of any regularized integral through the logarithm or the powers of the cutoff, the domain of validity of the regularized (DS) equations is not only restricted to the (UV) region, but can be extended to all range of the cutoff scale in contradistinction to dimensional regularization [3]. Thus this regularization scheme can give an insight into non-perturbative properties of the theory, even in the context of perturbative calculations where the (DS) equations appear as a resummation of perturbative series. For example this scheme is suitable for the calculation of the vacuum expectation value of operators like the condensates. Moreover, as this method is continuous and works strictly in four dimensions, it can be applied to the treatment of the axial anomaly which is known to connect both the (UV) and the (IR) regions.

In order to show that the partition function of (QED) is indeed finite, we derive first the regularized Dyson-Schwinger (DS) equations which are the basic ingredients for the skeleton expansion. The class of regularization is imposed by the choice of a cutoff function. We choose to mimic Schwinger's proper time regularization. Then, despite the fact that our regularization is non-perturbative, we deduce the regularized form of the relevant one particle irreducible (1PI) functions which are known to be potentially divergent by power counting² perturbatively from the regularized (DS) equations. We show that we recover the known result for the mass operator. This is also true for the vertex function if we assume that the electron lines are on mass-shell. Since the regularization of the path integral is gauge invariant by construction, the polarization operator is transverse and regularized. The finite part of the polarization operator has the standard form, the only discrepancy occurs in its

²The direct study of the regularized partition function with group renormalization techniques will be done in a forthcoming paper.

divergent part where a quadratic divergence appears. This divergent term can be removed by adding to the lagrangian a gauge invariant counterterm S_Λ which is quadratic in the photon field. The paper is organized as follows. In section 2 we define the regularized gauge invariant action in configuration and in momentum space. The regularized form of the (DS) equations and of the (WT) identity are given too. We derive the relevant (1PI) functions from the equations of motion in the next sections. Section 3 is devoted to the polarization operator and to the 2n-photon amplitudes. In addition to this section the role of S_Λ is clarified. We calculate the electron mass operator and the vertex function in section 4.

2 The regularized gauge invariant action, the (DS) equations and the (WT) identity

In order to regularize the action of QED in four dimensions we construct a non-local cutoff action which converges asymptotically to the standard non-regularized one as the cutoff tends to infinity. This can be done by a suited smearing of the point-like interactions of the fields by the scalar cutoff functions

$$\rho_i(x, y) = \int d\bar{k} e^{-ik(x-y)} \rho_i(k, \Lambda), \quad (2.1)$$

which are symmetrical and are regularized forms of Dirac's δ function, i.e.

$$\lim_{\Lambda \rightarrow \infty} \rho_i(k, \Lambda) = 1, \quad (2.2)$$

$$\lim_{\Lambda \rightarrow \infty} \rho_i(x, y) = \delta(x - y). \quad (2.3)$$

Here and in the following we work in Minkowski space. We choose the signature of the metric to be $(1, -1, -1, -1)$ and the notation $dx \equiv d^4x$ and $d\bar{k} \equiv d^4k/(2\pi)^4$.

In all analytically methods used to calculate or to define the Greens's functions of a given theory the inverse of the energy kinetic terms appear through the propagators. This is the case for non-perturbative methods based on exact renormalization group equation or Dyson-Schwinger expansion based on the use of the equations of motion of the theory and or pure perturbative methods based on Feynman diagrams expansion. It follows that the regularized action S_{Reg} of QED must obey three conditions.

(1) S_{Reg} must be gauge invariant.

(2) In any expression composed of product of (1PI) functions the inverse of the cutoff functions (2.1) associated to the kinetic terms which occur in the propagators must not cancel the cutoff functions associated to the vertices.

(3) When the cutoff Λ goes to infinity S_{Reg} must converge to the local non-regularized action of QED.

Since in QED the photon has no self-interaction we need only to regularize the fermionic part of the action. A solution to these constraints is to regularize separately in a gauge invariant manner the kinetic, the interaction and the mass terms. The minimal solution is to keep the mass term unregularized. Then the building blocks needed to construct S_{Reg} are the smeared gauge field

$$\mathbf{A}^\mu(\mathbf{x}) = \int d\mathbf{y} \rho_3(\mathbf{x}, \mathbf{y}) \mathbf{A}^\mu(\mathbf{y}), \quad (2.4)$$

the functional of the smeared gauge field

$$L(z, z') = \int dy C_\mu(z, z', y) \mathbf{A}^\mu(\mathbf{y}) \quad (2.5)$$

and the functionals

$$\Psi_{\mathbf{i}}(\mathbf{x}) = \int d\mathbf{y} \rho_{\mathbf{i}}(\mathbf{x}, \mathbf{y}) e^{i\mathbf{e} \cdot \mathbf{L}(\mathbf{x}, \mathbf{y})} \psi(\mathbf{y}) \quad (2.6)$$

of the smeared fermion and gauge fields which remembers Schwinger's point splitting [5]. Likewise in lattice gauge theory the functional $L(z, z')$ plays the role of a link. The form of the C-number function C_μ is imposed by the necessity of gauge covariance. In order that the functional $\Psi_{\mathbf{i}}(\mathbf{x})$ transforms covariantly under the gauge transformation

$$\begin{aligned} A_\mu(x) &\rightarrow A_\mu(x) - \frac{1}{e} \partial_\mu \Lambda(x) \\ \psi(x) &\rightarrow e^{i \int dy \rho_3(x, y) \Lambda(y)} \psi(x), \end{aligned} \quad (2.7)$$

the variation of $L(z, z')$ must be

$$\delta L(z, z') = \frac{1}{e} \int dy (\rho_3(z, y) - \rho_3(z', y)) \Lambda(y). \quad (2.8)$$

This imposes that the divergence of the C-number function C_μ verifies

$$\partial^\mu C_\mu(z, z', x) = \delta(x - z) - \delta(x - z'). \quad (2.9)$$

In addition the choice of C_μ is such that

$$L(z, z) = 0. \quad (2.10)$$

This is the necessary condition to recover the local fundamental fields from (2.4) and (2.6) when the cutoff tends to infinity. In fact due the properties (2.3) of the cutoff functions we have now,

$$\begin{aligned} \lim_{\Lambda \rightarrow \infty} \mathbf{A}^\mu(\mathbf{x}) &= \mathbf{A}^\mu(\mathbf{x}) \\ \lim_{\Lambda \rightarrow \infty} \Psi_{\mathbf{i}}(\mathbf{x}) &= \psi(\mathbf{x}). \end{aligned} \quad (2.11)$$

For C_μ we choose the less (UV) divergent solution

$$C_\mu(z, z', x) = i \int d\bar{r} \frac{r_\mu}{r^2} \left(e^{-ir(x-z)} - e^{-ir(x-z')} \right). \quad (2.12)$$

Then the regularized action S_{Reg} of QED is the sum of the fermionic term $S_{Fermion}$ and of the pure gauge term S_{Gauge} which are defined by

$$\begin{aligned}
S_{Fermion} = & \int dx dz dz' \bar{\psi}(z) e^{ieL(z,x)} \left[\rho_1(z,x) \left(i\partial + e(\partial L(x,z')) \right) \rho_1(x,z') \right. \\
& - e\rho_2(z,x) \left(\mathbf{A}(\mathbf{x}) + (\partial \mathbf{L}(\mathbf{x}, \mathbf{z}')) \right) \rho_2(\mathbf{x}, \mathbf{z}') \left. \right] e^{ie\mathbf{L}(\mathbf{x}, \mathbf{z}')} \psi(\mathbf{z}') \\
& - m \int dx \bar{\psi} \psi
\end{aligned} \tag{2.13}$$

$$S_{Gauge} = - \int dx \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \right] + S_\Lambda. \tag{2.14}$$

The contribution S_Λ of (2.14) is gauge invariant and will play the role of a counterterm for the polarization operator. This term will be discussed in more details in section 3. Since $L(z, z')$ is defined by (2.5) and (2.12), the partial derivative $\partial L(x, z')$ which enters in (2.13) is a function of x only. This implies that the kinetic and interaction terms of the regularized action (2.13) have the general form $\int dx \bar{\Psi}_i \mathbf{K}_i \Psi_i$, where \mathbf{K}_i is a functional of the smeared gauge field (2.4). As a result one can readily check that each of the three terms of (2.13), which correspond respectively to the kinetic, the interaction and the mass term, are by construction separately gauge invariant. When the cutoff Λ tends to infinity, it follows from the relations (2.11) that the terms proportional to $\partial L(x, z')$ cancel and that the regularized action $S_{Reg} - S_\Lambda$ converges formally to the standard local action of QED. In deriving the relations (2.11) and the asymptotic form of the regularized action we have assumed that we can formally interchange the limit with the integral symbol. This assumption is not exactly true³, and will be to the origin of the appearance of a quadratic divergence in the polarization operator. If we use the commutation relation

$$[i\partial, e^{ieL(x,z')}] = -e e^{ieL(x,z')} (\partial L(x, z')), \tag{2.15}$$

we can rewrite (2.13) in a more concise form as

$$\begin{aligned}
S_{Fermion} = & \int dx dz dz' \bar{\psi}(z) e^{ieL(z,z')} \left[\rho_1(z,x) i\partial \rho_1(x,z') \right. \\
& - e\rho_2(z,x) \left(\mathbf{A}(\mathbf{x}) + (\partial \mathbf{L}(\mathbf{x}, \mathbf{z}')) \right) \rho_2(\mathbf{x}, \mathbf{z}') \left. \right] \psi(\mathbf{z}') \\
& - m \int dx \bar{\psi} \psi.
\end{aligned} \tag{2.16}$$

From now on, the regularized gauge invariant path integral which we consider in configuration space is

$$Z(\eta, \bar{\eta}, J) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_\mu e^{i(S(\psi, \bar{\psi}, A) + \int dx (\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x) + J_\mu(x)A^\mu(x)))} \tag{2.17}$$

³Even in one dimension this is not in general the case for improper integral.

where $S(\psi, \bar{\psi}, A)$ is the sum of (2.14) and (2.16). The η , $\bar{\eta}$ and J_μ are respectively the sources for the fermion and gauge fields. As before one can notice that the new form (2.16) of the fermionic part of the action is still composed of three terms which are separately gauge invariant. The fact that the whole action $S(\psi, \bar{\psi}, A)$ is a regularized action of QED in four dimensions will become transparent in momentum space where we give the representation of the partition function (2.17) and of the Legendre transform of the (1PI) generating functional

$$G(\eta, \bar{\eta}, J) \equiv \log Z(\eta, \bar{\eta}, J). \quad (2.18)$$

For the partition function we have

$$Z(\eta, \bar{\eta}, J) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_\mu e^{i(S(\psi, \bar{\psi}, A) + \int d\bar{k} (\bar{\eta}(k)\psi(k) + \bar{\psi}(k)\eta(k) + J_\mu(-k)A^\mu(k))}. \quad (2.19)$$

In this expression the action is

$$S(\psi, \bar{\psi}, A) = \int d\bar{k} d\bar{p} d\bar{p}' \bar{\psi}(p) K(k, p, p') \psi(p') + S_{Gauge} \quad (2.20)$$

$$S_{Gauge} = -\frac{1}{2} \int d\bar{k} \left(k^2 g_{\mu\nu} - k_\mu k_\nu \left(1 - \frac{1}{\xi}\right) \right) A^\mu(k) A^\nu(-k) + S_\Lambda, \quad (2.21)$$

and the kernel K is given by

$$K(k, p, p') = F(p - k, k - p') \rho_1^2(k) \not{k} - m((2\pi)^4)^2 \delta(p - k) \delta(p' - k) \\ - e \int d\bar{q} \rho_3(q) A^\mu(q) \Gamma_\mu(q) F(p - k, k - q - p') \rho_2(k) \rho_2(k - q), \quad (2.22)$$

where the transverse matrix $\Gamma_\mu(q)$ ⁴ is defined by

$$\Gamma_\mu(q) = \gamma_\mu - \not{q} \frac{q_\mu}{q^2}, \quad (2.23)$$

and $F(p, q)$ is the Fourier transform of $e^{ieL(x, z')}$. With the help of the recursion formulas of appendix B which define $F(p, q)$, we can rewrite the action (2.20) in the final form

$$S(\psi, \bar{\psi}, A) = \int d\bar{k} \bar{\psi}(k) \left(\rho_1^2(k) \not{k} - m \right) \psi(k) - e \int d\bar{p} d\bar{p}' \bar{\psi}(p) A^\mu(p - p') \Gamma_\mu(p, p') \psi(p') \\ + \sum_{n=0}^{+\infty} \frac{(ie)^{(n+2)}}{(n+2)!} \int d\bar{k} d\bar{p} d\bar{p}' \bar{\psi}(p) \left[F_{n+2}(p - k, k - p') \rho_1^2(k) \not{k} + i(n+2) \right. \\ \left. \int d\bar{q} \rho_3(q) A^\mu(q) \Gamma_\mu(q) F_{n+1}(p - k, k - q - p') \rho_2(k) \rho_2(k - q) \right] \psi(p') \\ + S_{Gauge}. \quad (2.24)$$

Here the matrix $\Gamma_\mu(p, p')$ is defined in terms of (2.23) by

$$\Gamma_\mu(p, p') = \rho_3(p - p') \left[\rho_2(p) \rho_2(p') \Gamma_\mu(p - p') + \frac{(p - p')_\mu}{(p - p')^2} \left(\rho_1^2(p) \not{p} - \rho_1^2(p') \not{p}' \right) \right], \quad (2.25)$$

⁴See the properties of this matrix in appendix A.

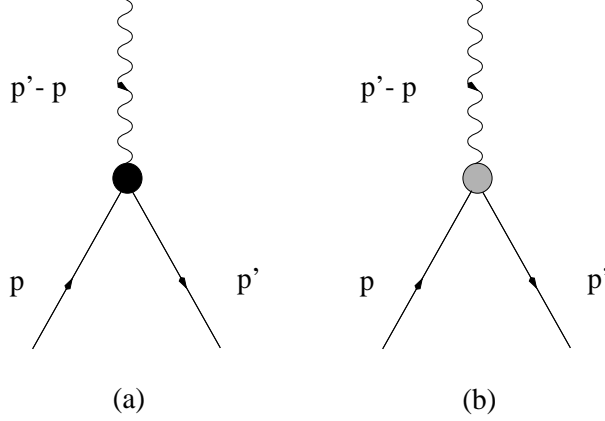


Figure 1: The vertex associated to the matrix $-e\Gamma_\mu(p, p')$ (a) and to its transversal part (b).

and has the property

$$\lim_{\Lambda \rightarrow \infty} \Gamma_\mu(p, p') = \gamma_\mu. \quad (2.26)$$

Owing to the property (2.26) the first and the second term of (2.24) converge to the standard fermionic part of the action of QED when the cutoff goes to infinity. As for the third term, it vanishes identically due to the properties (B.6) of the F_n . This term which describes an infinite set of vertices is needed to ensure the gauge invariance of the regularized amplitudes. Therefore the whole action (2.24) converges formally to the non-regularized action of QED. Since the third term of (2.24) vanishes asymptotically, we expect that the matrix $-e\Gamma_\mu(p, p')$ (2.25) or its transversal part $-e\rho_3(p - p')\rho_2(p)\rho_2(p')\Gamma_\mu(p - p')$ associated to the vertices of Fig.1 will play an essential role in the representation of amplitude by Feynman diagrams. Notice that the matrix $\Gamma_\mu(p, p')$ (2.25) is unambiguously defined by the values of the momentas of the two electrons lines because $\Gamma_\mu(p, p')$ is symmetrical in the variables p and p' .

From the action (2.24) it is readily seen that the free electron and photon propagators behave respectively for high k^2 like $1/m$ and $1/k^2$. Since at each vertex is associated a cutoff function, it follows that the path integral (2.19) defined by the action (2.24) is finite. Apart from the fact that in euclidian space the (UV) cutoff functions must be rapid decreasing functions of the squared momenta and must be defined for all values of the momentum, the choice of their form is quite arbitrary. For instance if all the cutoff functions (2.1) are identical the action (2.24) which is expressed in terms of the fundamental fields does not simplify. However when expressed in configuration space in terms of the smeared fields (2.4) and (2.6) the fermionic part of the regularized action take the compact form

$$S_{Fermion} = \int dx \bar{\Psi} i \not{D} \Psi - m \int d\mathbf{x} \bar{\psi} \psi, \quad (2.27)$$

where $i\mathcal{D}$ is the smeared covariant derivative defined by

$$i\mathcal{D} = i\partial - e\mathcal{A}. \quad (2.28)$$

From a practical point of view it is better to leave the cutoff functions unconstrained, then in the following we consider the general action (2.24). In order to check that the theory is indeed finite, we shall derive in the next sections the regularized (1PI) functions from the regularized equations of motion.

For this purpose we also work out the Legendre transform of (2.18) in momentum space. It is given in terms of the vacuum expectation value of the fields

$$\bar{\psi}(k) = i(2\pi)^4 \frac{\delta G}{\delta \eta(k)} \quad \psi(k) = -i(2\pi)^4 \frac{\delta G}{\delta \bar{\eta}(k)} \quad A_\mu(-k) = -i(2\pi)^4 \frac{\delta G}{\delta J^\mu(k)} \quad (2.29)$$

by

$$\Gamma(\psi, \bar{\psi}, A) = -iG(\eta, \bar{\eta}, J) - \int d\bar{k} (\bar{\eta}(k)\psi(k) + \bar{\psi}(k)\eta(k) + J_\mu(-k)A^\mu(k)). \quad (2.30)$$

We recall the conjugated relations for the sources

$$\eta(k) = -(2\pi)^4 \frac{\delta \Gamma}{\delta \bar{\psi}(k)} \quad \bar{\eta}(k) = (2\pi)^4 \frac{\delta \Gamma}{\delta \psi(k)} \quad J_\mu(-k) = -(2\pi)^4 \frac{\delta \Gamma}{\delta A^\mu(k)}. \quad (2.31)$$

From the invariance of the partition function (2.19) under infinitesimal translation of the fermion and gauge fields and under infinitesimal gauge transformation we deduce respectively the (DS) equations and the (WT) identities which are now mathematically well-defined objects. The regularized form of the (DS) equations is

$$0 = \left\langle \int d\bar{k} d\bar{p} d\bar{p}' \bar{\psi}(p) \frac{\delta K(k, p, p')}{\delta A^\nu(r)} \psi(p') - \left(r^2 g_{\mu\nu} - r_\mu r_\nu \left(1 - \frac{1}{\xi}\right) \right) \frac{A^\mu(-r)}{(2\pi)^4} + \frac{J_\nu(-r)}{(2\pi)^4} + \frac{\delta S_\Lambda}{\delta A^\nu(r)} \right\rangle \quad (2.32)$$

$$0 = \left\langle \int d\bar{k} d\bar{p}' K(k, p, p') \psi(p') + \eta(p) \right\rangle, \quad (2.33)$$

where the kernel $K(k, p, p')$ is explicitly and implicitly given by the expression of the action (2.22) or (2.24) respectively. Due to the relations (2.31) the regularized expression of the (WT) in terms of the generating functional of (1PI) functions can be written in momentum space as

$$(2\pi)^4 r^\mu \frac{\delta \Gamma}{\delta A^\mu(r)} = e\rho_3(r) \int dk \left[\bar{\psi}(k) \frac{\delta \Gamma}{\delta \bar{\psi}(k-r)} + \frac{\delta \Gamma}{\delta \psi(k)} \psi(k-r) \right] - \frac{1}{\xi} r^2 r^\mu A_\mu(-r). \quad (2.34)$$

Except for the additional cutoff function ρ_3 the above expression is identical to the standard one. Thus we expect that all general relations among Green's functions which can be deduced from (2.34) are the same as in the case of known regularization schemes.

3 The polarization operator and the 2n-photon amplitudes

In many gauge invariant regularization schemes the calculation of amplitudes relative to closed fermion loops is a test of the method. In fact any contribution to these amplitudes must be unambiguously finite. We will see that this is indeed the case for the polarization operator and that due to explicit gauge invariance of the method this conclusion remains valid for the 2n-photon amplitudes.

In order to calculate the polarization operator we take the functional derivative of the equation of motion (2.32) with respect to the C-number function $A^\mu(-r')$. Then using (2.31) to express $J^\nu(-r)$, the inverse of the full photon propagator (C.3) is given in terms of the polarization operator $\pi_{\mu\nu}$ by

$$(2\pi)^4 \frac{\delta^2 \Gamma}{\delta A^\mu(-r') \delta A^\nu(r)} = \pi_{\mu\nu}(r, r') - \delta(r - r') \left(r^2 g_{\mu\nu} - r_\mu r_\nu (1 - \frac{1}{\xi}) \right) + (2\pi)^4 \frac{\delta}{\delta A^\mu(-r')} \langle \frac{\delta S_\Lambda}{\delta A^\nu(r)} \rangle + \mathcal{O}(\mathcal{A}) \quad (3.1)$$

$$\pi_{\mu\nu}(r, r') = (2\pi)^4 \frac{\delta}{\delta A^\mu(-r')} \int d\bar{k} d\bar{p} d\bar{p}' \langle \bar{\psi}(p) \frac{\delta K(k, p, p')}{\delta A^\nu(r)} \psi(p') \rangle. \quad (3.2)$$

Here $\mathcal{O}(\mathcal{A})$ represents terms which are vanishing when the sources are switched off. We will now calculate the right hand-side of (3.2) perturbatively at the one loop level. The comparison of this result with the known expression of the regularized form of $\pi_{\mu\nu}$ will then constrain the form of the operator S_Λ . Since in the action (2.24) each monomial is proportional to the product of n gauge field⁵ and is of order e^n , at order e^2 , the operator $\bar{\psi}(\delta K / \delta A^\nu) \psi$ is at most linear in the gauge field. Hence using (2.20) and (2.24) its vacuum expectation value can be expressed in terms of the full electron propagator

$$S(p', p) = -i(2\pi)^4 \frac{\delta^2 G}{\delta \bar{\eta}(p') \delta \eta(p)}, \quad (3.3)$$

which is defined in presence of external sources, and only in terms of the vacuum expectation value of the gauge field.

Knowing that the functional derivative of S with respect to $A^\mu(-r')$ is at least of order e , we obtain for $\pi_{\mu\nu}$

$$\begin{aligned} \pi_{\mu\nu}(r, r') = & (2\pi)^4 Tr \left\{ ie \int d\bar{p} \Gamma_\nu(p, p - r) \frac{\delta S(p - r, p)}{\delta A^\mu(-r')} \right. \\ & - \frac{1}{2} e^2 \int d\bar{k} d\bar{p} d\bar{p}' \left[\frac{r'_\mu}{r'^2} \rho_3(r') \left(\frac{\delta F_1}{\delta A^\nu(r)}(p - k + r', k - p') \right) \right. \\ & - \frac{\delta F_1}{\delta A^\nu(r)}(p - k, k - p' + r') \left. \right] \rho_1^2(k) \not{k} + \rho_2(k) \rho_2(k + r') \rho_3(r') \Gamma_\mu(r') \\ & \left. \frac{\delta F_1}{\delta A^\nu(r)}(p - k, k + r' - p') + (r' \leftrightarrow -r, \mu \leftrightarrow \nu) \right] S(p', p) \left. \right\} \end{aligned} \quad (3.4)$$

⁵See the expression of F in appendix B.

If $\Gamma^{(2)}$ is the inverse of the full electron propagator, we can express $\frac{\delta S}{\delta A^\mu}$ in terms of the three point function $\frac{\delta \Gamma^{(2)}}{\delta A^\mu}$ (C.5), and using (B.4) and (B.5) we get at order e

$$\frac{\delta S(p', p)}{\delta A^\mu(-r')} = \delta(p' - p + r') \frac{e}{(2\pi)^4} S(p') \Gamma_\mu(p', p) S(p). \quad (3.5)$$

Here $\Gamma_\mu(p', p)$ is defined in (2.25) and $S(p)$ is the free electron propagator whose expression is readily deduced from the first term of (2.24) and is given by

$$S(p) = \frac{1}{\rho_1^2(p) \not{p} - m}. \quad (3.6)$$

Due to the properties (2.26) and (B.6) we can notice that the expression (3.4) converges formally to the non regularized form of the polarization operator when the cutoff Λ goes to infinity.

Now if we substitute in (3.4) $\delta S / \delta A^\mu$ by its expression (3.5), we can see that the polarization operator

$$\pi_{\mu\nu}(r) \delta(r - r') \equiv \pi_{\mu\nu}(r, r') \quad (3.7)$$

has the following structure

$$\pi_{\mu\nu}(r) = \text{Tr} [r_\mu r_\nu A + (r_\mu \Gamma_\nu(r) B_1 + r_\nu \Gamma_\mu(r) B_2) + \Gamma_\mu(r) \otimes \Gamma_\nu(r) C]. \quad (3.8)$$

Since the theory is gauge invariant by construction, the (WT) (2.34) tells us that the polarization operator must be transverse. Owing to the transversality property (A.2) of the Γ_μ matrices, this implies that the matrices A , B_1 and B_2 of (3.8) must be zero if we multiply both sides of (3.8) successively by $r^\mu r^\nu$, and then by r^μ and r^ν . This is indeed the case. A straightforward calculation shows that B_1 and B_2 are both proportional to the integral

$$\int dp \rho_2(p) (\rho_2(p+r) S(p) - \rho_2(p-r) S(p-r)), \quad (3.9)$$

and that the matrix A is proportional to the integral

$$\int dp \left[(\rho_1^2(p) \not{p} - m) S(p-r) - (\rho_1^2(p+r) (\not{p} + \not{r}) - m) S(p) \right]. \quad (3.10)$$

Since for high squared momenta the asymptotic form of the free electron propagator (3.6) is $1/m$ the two integrals (3.9) and (3.10) are regularized and vanish identically after a shift of variable. However there is a great difference between the matrix A and the matrices B_1 and B_2 . Whereas in the massless case one can always choose the cutoff functions in order to regularize the matrices B_1 and B_2 given by (3.9), this is not true for A where now terms like $\rho_1^2(p) \rho_1^{-2}(p-r)$ appear in the integral (3.10). In contradistinction to the massive case, it seems very difficult to construct a continuous non-perturbative regularization scheme for massless fermions which is also mathematically clean. This is reminiscent of the Nielsen-Ninomiya Theorem [15] which implies that a chiral invariant regularization is still lacking in the fermionic sector of lattice gauge theory.

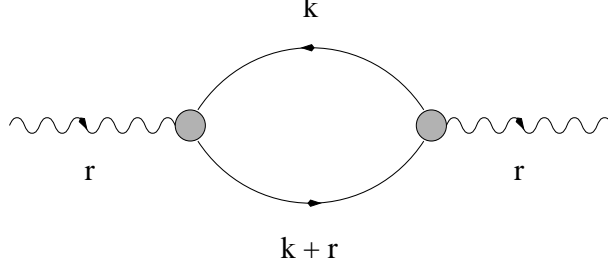


Figure 2: The polarization operator.

We thus obtain for the regularized form of the polarization operator (3.2)

$$\pi_{\mu\nu}(r) = i \frac{e^2}{(2\pi)^4} \rho_3^2(r) \text{Tr} \int dk \left[\rho_2^2(k) \rho_2^2(k+r) \Gamma_\mu(r) S(k+r) \Gamma_\nu(r) S(k) \right]. \quad (3.11)$$

The polarization operator can be represented by the diagram Fig.2 where all the vertices are transverse Fig.1b. In the present work we calculate the relevant (1PI) functions with cutoff functions of the form

$$\rho(k) = e^{\frac{k^2}{\Lambda^2}}, \quad (3.12)$$

Λ being the (UV) cutoff. In order to simplify the calculation we choose,

$$\rho_1^4(k) = \rho_2(k), \quad \rho_2(k) = \rho_3(k) = \rho(k). \quad (3.13)$$

In this case we can easily show that

$$\int dk f(k, \Lambda) S(k) = \int dk f(k, \Lambda) \rho^{-1}(k) \frac{\sqrt{\rho(k)} k + m}{k^2(1 + \frac{m^2}{\Lambda^2}) - m^2} + \mathcal{O}(\frac{\infty}{*}) \quad (3.14)$$

provided that $|f(k, \Lambda)|$ behaves like $\rho^2(k)/k$ for high k^2 . This implies that we can always make the replacement

$$S(k) \rightarrow (1 - \frac{m^2}{\Lambda^2}) \rho^{-1}(k) \frac{\sqrt{\rho(k)} k + m}{k^2 - m^2} \quad (3.15)$$

in the integrals which we encounter in the calculation of the relevant (1PI) functions without changing the result of the integration.

If we take the trace after the substitution (3.15) and use the Feynman parametrization for the product of propagators the expression (3.11) can be written as

$$\begin{aligned} \pi_{\mu\nu}(r) = & 4i \rho^2(r) \frac{e^2}{(2\pi)^4} \frac{1}{r^4} (1 - 2 \frac{m^2}{\Lambda^2}) \int_0^1 dx \int dk \frac{1}{[-k^2 - 2krx + m^2 - r^2x]^2} \left\{ e^{\frac{3}{2} \frac{k^2}{\Lambda^2}} e^{\frac{3}{2} \frac{(k+r)^2}{\Lambda^2}} \right. \\ & 2 \left[(kr)^2 r_\mu r_\nu - r^2 kr (r_\mu k_\nu + r_\nu k_\mu) + r^4 k_\mu k_\nu - \frac{r^2}{2} (r^2 g_{\mu\nu} - r_\mu r_\nu) (k^2 + kr) \right] \\ & \left. + e^{\frac{k^2}{\Lambda^2}} e^{\frac{(k+r)^2}{\Lambda^2}} m^2 r^2 (r^2 g_{\mu\nu} - r_\mu r_\nu) \right\} \end{aligned} \quad (3.16)$$

The asymptotic form of the integrals which contribute to the right hand-side of (3.16) are given in appendix D. With the results (D.7) and (D.8) and if we introduce an arbitrary unit of mass μ , (3.16) can be rewritten

$$\begin{aligned} \pi_{\mu\nu}(r) = & -\frac{\alpha}{\pi}(r^2 g_{\mu\nu} - r_\mu r_\nu) \left\{ \frac{1}{2r^2} \left[\frac{\Lambda^2}{3} - m^2 \left(\frac{5}{3} - 2 \log \frac{3}{2} \right) \right] + \frac{2}{3} \log \frac{\Lambda}{\mu} - \frac{\gamma}{3} + \frac{29}{72} - \frac{1}{3} \log 3 \right. \\ & \left. - 2 \int_0^1 dx \, x(1-x) \log \frac{m^2 - r^2 x(1-x)}{\mu^2} \right\}, \end{aligned} \quad (3.17)$$

α being the fine structure constant. We see from the expression (3.17) that the polarization operator is transverse as expected and has the standard form [17], except for the quadratic divergent term which is proportional to $1/r^2$. As noticed in Section 2 the origin of this term is due to the fact that in configuration space the regularized action $S_{Reg} - S_\Lambda$ (2.13) and (2.14) converges exactly to the non-regularized action of QED if one makes the assumption that taking the limit commutes with integration. Stated differently, this discrepancy is due to the non-uniform convergence of the integral in momentum space. The new divergent term of (3.17) can be removed by adding the counterterm

$$S_\Lambda = \frac{\alpha}{4\pi} \int d\bar{k} \, \rho_3^2(k) \frac{1}{k^2} (k^2 g_{\mu\nu} - k_\mu k_\nu) \left[\frac{\Lambda^2}{3} - m^2 \left(\frac{5}{3} - 2 \log \frac{3}{2} \right) \right] A^\mu(k) A^\nu(-k) \quad (3.18)$$

to the expression of the regularized action S_{Reg} in momentum space. The counterterm S_Λ can be viewed as associated to a new bare interaction

$$S_C = c \frac{\alpha}{4\pi} \int d\bar{k} \, \rho_3^2(k) \frac{1}{k^2} (k^2 g_{\mu\nu} - k_\mu k_\nu) A^\mu(k) A^\nu(-k), \quad (3.19)$$

where the coupling constant c has the dimension of the square of a mass. This interaction which is non-local in configuration space can be absorbed in the inverse of the photon propagator (3.1). Since at order α^n the photon propagator (C.3) will now contain a term proportional to $c^n/(k^2)^{(n+1)}$, the contributions of this interaction to the potentially divergent (1PI) functions are all (UV) finite. This implies that the new coupling constant c , which can give masses to the photon, is not renormalized by higher order corrections. It follows that the counterterm S_Λ is uniquely defined by the choice of the renormalized form of the coupling constant c . In order to keep the photon massless as in standard QED and to avoid any non-local interaction in the action, we restrict ourselves to the form (3.18) of the counterterm, which means that the renormalized form of the coupling constant c is tuned to zero. This gives the known result for the polarization operator (3.2). Another choice for the bare coupling constant c , that is to say for the counterterm S_Λ , seems to open the possibility of dynamical mass generation for the photon through the non-local interaction (3.19).

Now we show that the expressions for the 2n-photon amplitudes are standard. We begin by the calculation of the polarization tensor. If we use the equation of motion (2.32) and

the definitions (2.31), since S_Λ is quadratic in the gauge field, the (1PI) function relative to the four photons amplitudes is given by

$$\frac{\delta^4 \Gamma}{\delta A^\beta(r_4) \delta A^\alpha(r_3) \delta A^\nu(r_2) \delta A^\mu(r_1)} = -i(2\pi)^4 \frac{\delta^3}{\delta A^\beta(r_4) \delta A^\alpha(r_3) \delta A^\nu(r_2)} Tr \int d\bar{k} d\bar{p} d\bar{p}' \frac{\delta K}{\delta A^\mu(r_1)}(k, p, p', \frac{\delta}{\delta J}) S(p', p) e^G + \mathcal{O}(\mathcal{A}), \quad (3.20)$$

where the differential operator $\delta K/\delta A^\mu$ with respect to the gauge field sources J is easily obtained from (2.20), (2.24), (B.3) and (B.4).

In the last expression it is understood that $\delta K/\delta A$ acts first on the product Se^G and then that we act successively with the functionals derivatives $\delta/\delta A$. Knowing that the action of $\delta/\delta J$ or $\delta/\delta A$ on S (3.3) gives rise to terms which are at least of order e^6 and that at the one loop order terms of the order $\mathcal{O}(\nabla)$ are neglected the polarization tensor

$$\Gamma_{\mu\nu\alpha\beta}^{(4)}(r_1, r_2, r_3, r_4) \delta(r_1 + r_2 + r_3 + r_4) \equiv ((2\pi)^4)^3 \frac{\delta^4 \Gamma}{\delta A^\beta(r_4) \delta A^\alpha(r_3) \delta A^\nu(r_2) \delta A^\mu(r_1)} \quad (3.21)$$

can be written as,

$$\begin{aligned} \Gamma_{\mu\nu\alpha\beta}^{(4)}(r_1, r_2, r_3, r_4) = & Tr \left[r_{\mu 1} r_{\nu 2} r_{\alpha 3} r_{\beta 4} A + (r_{\nu 2} r_{\alpha 3} r_{\beta 4} \Gamma_\mu(r_1) B_1 + \text{three similar}) \right. \\ & + (r_{\alpha 3} r_{\beta 4} \Gamma_\mu(r_1) \otimes \Gamma_\nu(r_2) C_1 + \text{five similar}) \\ & + (r_{\beta 4} \Gamma_\mu(r_1) \otimes \Gamma_\nu(r_2) \otimes \Gamma_\alpha(r_3) D_1 + \text{three similar}) \\ & \left. + \Gamma_\mu(r_1) \otimes \Gamma_\nu(r_2) \otimes \Gamma_\alpha(r_3) \otimes \Gamma_\beta(r_4) E \right]. \end{aligned} \quad (3.22)$$

Here A , B_i , C_i , D_i and E are scalar functions of the four momenta r_1 , r_2 , r_3 and r_4 and tensorial expressions with respect to the spin indices. Since the regularized polarization tensor is gauge invariant by construction, it follows that A is identical to zero if both sides of (3.22) are multiplied by the product $r_1^\mu r_2^\nu r_3^\alpha r_4^\beta$. Similarly we can show that only the scalar function E in (3.22) is non vanishing. The structure of $\Gamma_{\mu\nu\alpha\beta}^{(4)}$ being now established, we keep only the term proportional to Γ_μ in $\delta K/\delta A$. Since this term is not a differential operator the calculation of the right member of (3.21) is indeed reduced to the evaluation of $\delta/\delta A^\beta \delta/\delta A^\alpha \delta/\delta A^\nu S$. If we define

$$\begin{aligned} A_{\mu\nu\alpha\beta}(r_1, r_2, r_3, r_4) \equiv & i \frac{e^4}{(2\pi)^4} \rho_3(r_1) \rho_3(r_2) \rho_3(r_3) \rho_3(r_4) Tr \int dk \rho_2^2(k) \rho_2^2(k - r_1) \\ & \rho_2^2(k - r_1 - r_2) \rho_2^2(k + r_4) \left[\Gamma_\mu(r_1) S(k - r_1) \Gamma_\nu(r_2) S(k - r_1 - r_2) \right. \\ & \left. \Gamma_\alpha(r_3) S(k + r_4) \Gamma_\beta(r_4) S(k) \right] \end{aligned} \quad (3.23)$$

and use the result (3.22) and the definitions (3.13) for the cutoff functions we find for the polarization tensor Fig.3

⁶See appendix C.

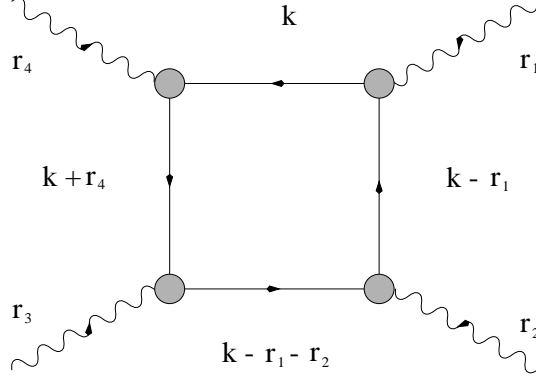


Figure 3: The representation of the amplitude $A_{\mu\nu\alpha\beta}(r_1, r_2, r_3, r_4)$.

$$\begin{aligned}
\Gamma_{\mu\nu\alpha\beta}^{(4)}(r_1, r_2, r_3, r_4) &= A_{\mu\nu\alpha\beta}(r_1, r_2, r_3, r_4) + A_{\mu\nu\beta\alpha}(r_1, r_2, r_4, r_3) + A_{\mu\beta\nu\alpha}(r_1, r_4, r_2, r_3) \\
&\quad + A_{\mu\alpha\nu\beta}(r_1, r_3, r_2, r_4) + A_{\mu\alpha\beta\nu}(r_1, r_3, r_4, r_2) \\
&\quad + A_{\mu\beta\alpha\nu}(r_1, r_4, r_3, r_2).
\end{aligned} \tag{3.24}$$

Now we are ready to show that $\Gamma_{\mu\nu\alpha\beta}^{(4)}$ is as expected finite [16] and is regular in the (IR) domain. After the substitution (3.15) and a judicious shift of the integration variable in (3.23), the product of the cutoff functions becomes proportional to $\rho^6(k)$, where $\rho(k)$ is defined in (3.12). From the rotational invariance of the integral and the properties (A.1) and (A.2) of the Γ_μ matrices, it follows that the asymptotic expression of (3.23) as $-q^2 \rightarrow +\infty$ is

$$A_{\mu\nu\alpha\beta} \sim \text{Tr} (2\Gamma_\mu \Gamma_\nu \Gamma_\alpha \Gamma_\beta - \Gamma_\alpha \Gamma_\mu \Gamma_\nu \Gamma_\beta - \Gamma_\beta \Gamma_\mu \Gamma_\alpha \Gamma_\nu) \int dk \rho^6(k) \frac{k^4}{(m^2 - k^2)^4}. \tag{3.25}$$

As a corollary it is readily seen that the potentially logarithmic divergent term of (3.24) vanishes identically in taking the sum, i.e. the polarization tensor is finite. Is the polarization tensor regular?

The standard one loop expression of the polarization tensor $\Gamma_{S\mu\nu\alpha\beta}^{(4)}$ which is known to be finite and gauge invariant [16], can be deduced from (3.23) and (3.24) if one substitutes respectively the matrices Γ_μ by the Dirac's matrices γ_μ . As a result, the structure of the difference $\Gamma_{\mu\nu\alpha\beta}^{(4)} - \Gamma_{S\mu\nu\alpha\beta}^{(4)}$ is given by (3.22) after the substitution of the matrices Γ_μ by Dirac's matrices γ_μ , if in this expression the last term in the right member is omitted. Since this difference is actually finite and gauge invariant, we can show along the same lines as before that it vanishes identically. Therefore the polarization tensor has the standard form, and thus is regular. Since for $n > 2$ the 2n-photon amplitudes are finite, we can show in a similar way that they are also regular. We end this section by discussing the status of the n-photon amplitudes when n is odd.

If S is the regularized electron propagator (3.6) and if the matrix Γ_μ is given by (2.23), the requirement of gauge invariance imposes that the n -photon amplitude, whose structure is similar to (3.22), must be proportional to the integral of the trace of n products of $\Gamma_\mu S$. Under charge conjugation the matrices Γ_μ transform exactly in the same way as Dirac's matrices γ_μ . Hence it follows by virtue of Furry's theorem that the n -photon amplitude vanishes when n is odd.

4 The mass operator and the vertex function

First we derive the expression of the mass operator at the one loop level.

For this sake we take the functional derivative of the equation of motion (2.33) defined by (2.20) and (2.24) with respect to the C-number function $\psi^b(p')$ and express the sources of the electron field according to the generating functional of (1PI) functions by relations (2.31). We thus obtain the inverse of the full electron propagator $\Gamma^{(2)}$ (C.1) in terms of the mass operator Σ as

$$\begin{aligned}\Gamma_{ab}^{(2)}(p, p') &= (\rho_1^2(p)\not{p} - m)_{ab}\delta(p - p') - \Sigma_{ab}(p, p') \\ \Sigma_{ab}(p, p') &= -ie \int dk dk' \Gamma_{db}^{(2)}(k', p') \Gamma_{ac}^\mu(p, k) \frac{\delta}{\delta J_\mu(k-p)} S_{cd}(k, k') e^G \\ &\quad - \sum_{n=0}^{+\infty} \frac{(ie)^{n+2}}{(n+2)!} \int d\bar{k} d\bar{k}' dp'' \Gamma_{db}^{(2)}(p'', p') \left[F_{n+2}(p - k, k - k') \rho_1^2(k) \not{k} \right. \\ &\quad \left. + (n+2) \int dq \rho_2(k) \rho_2(k - q) \rho_3(q) \Gamma_\mu(q) \frac{\delta}{\delta J_\mu(-q)} F_{n+1}(p - k, k - k' - q) \right]_{ac} \\ &\quad S_{cd}(k', p'') e^G + \mathcal{O}(\frac{\delta^2}{\delta^2}).\end{aligned}\tag{4.1}$$

Here the $F_n(p, q)$ which are defined in appendix B are expressed in terms of the functional derivative $\delta/\delta J$ and the latin letters refer to the spin indices. The contribution of order e^2 which comes from the bracket of expression (4.2) is proportional to $\Gamma_\mu(q) q_\nu D^{\mu\nu}(q)$ and vanishes owing to the properties (A.2). Taking into account the fact that the functional derivative $\delta S/\delta J_\mu$ can be written in terms of the product of the full photon propagator

$$D^{\mu\nu}(r, q) = i(2\pi)^4 \frac{\delta^2 G}{\delta J_\mu(r) \delta J_\nu(-q)}\tag{4.3}$$

times the functional derivative $\delta S/\delta A^\nu$ whose expression at order e is given by (3.5), we get for the mass operator at order e^2

$$\Sigma(p, p') = \delta(p - p') i \frac{e^2}{(2\pi)^4} \int dk D^{\mu\nu}(k - p) \Gamma_\mu(p, k) S(k) \Gamma_\nu(k, p).\tag{4.4}$$

This operator can be represented with the help of the vertex Fig.1a by the diagram of Fig.4. In this expression $D^{\mu\nu}(k)$ is the free photon propagator whose expression

$$D_{\mu\nu}(k) = -\frac{1}{k^2} \left(g_{\mu\nu} - \frac{1}{k^2} k_\mu k_\nu \left(1 - \frac{1}{\xi} \right) \right)\tag{4.5}$$

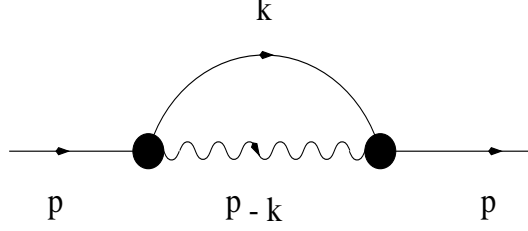


Figure 4: The mass operator.

is deduced from (2.21) and $\Gamma_\mu(p, k)$ is defined in (2.25). Owing to the property (2.26) of the matrix $\Gamma_\mu(p, k)$, we can notice that we recover the standard non regularized form for the mass operator as the cutoff Λ tends to infinity. The mass operator Σ can be written as

$$\Sigma(p)\delta(p - p') \equiv \Sigma(p - p') \quad (4.6)$$

$$\Sigma(p) = A(p^2) + \not{p}B(p^2). \quad (4.7)$$

As stated before, we can replace freely the free propagator $S(k)$ by the expression (3.15) without changing the result of integration in (4.4). Then if we use definitions (3.13) and (3.12) for the cutoff functions and neglect finite terms of the order $\mathcal{O}(\infty/*)$ we get for the scalar functions A and B

$$A(p^2) = -im\frac{e^2}{(2\pi)^4} \int dk \left\{ \rho^2(k-p) \frac{1}{[\kappa^2 - (k-p)^2]^2} + 3\rho^2(k-p)\rho(k) \frac{1}{[\kappa^2 - (k-p)^2](m^2 - k^2)} - \frac{p^2 - 2kp + m^2}{[\kappa^2 - (k-p)^2]^2(m^2 - k^2)} \right\} \quad (4.8)$$

$$p^2 B(p^2) = i\frac{e^2}{(2\pi)^4} \int dk \rho^2(k-p) \left\{ 2\rho^{\frac{3}{2}}(k) \frac{kp}{[\kappa^2 - (k-p)^2](m^2 - k^2)} + \left[kp \left(\rho^{\frac{3}{2}}(k) - \rho^{\frac{1}{2}}(k) \right) + 2p^2 \left(1 - \rho^{\frac{3}{2}}(k) \right) \right] \frac{1}{[\kappa^2 - (k-p)^2]^2} \right\} \quad (4.9)$$

where the last integral of the right member of (4.8) is finite. In order to avoid the problem of potentially (IR) divergences due to the momentum carried by the internal photon line, we give an arbitrary small mass κ to the photon. After Feynman's parametrization of the product of propagators, with the help of formulas (D.6), (D.7) and (D.9), the integration over the four momentum is easily performed and we obtain in the Feynman gauge the known results [17]

$$A(p^2) = m\frac{\alpha}{\pi} \left\{ 2 \log \frac{\Lambda}{\mu} - \gamma - 1 - \frac{1}{4} \log 54 - \int_0^1 dx \log \frac{-p^2 x(1-x) + m^2(1-x) + x\kappa^2}{\mu^2} \right\} \quad (4.10)$$

$$B(p^2) = -\frac{\alpha}{2\pi} \left\{ \log \frac{\Lambda}{\mu} - \frac{\gamma}{2} - \frac{73}{140} - \frac{1}{2} \log \frac{8}{5} - \int_0^1 dx x \log \frac{-p^2 x(1-x) + m^2(1-x) + x\kappa^2}{\mu^2} \right\}. \quad (4.11)$$

We now calculate the vertex function at the one loop level. If we define the (1PI) function $\Gamma^{(3)}$ as,

$$((2\pi)^4)^2 \frac{\delta^3 \Gamma}{\delta A^\mu(-r) \delta \psi(p) \delta \bar{\psi}(p')} = \Gamma_\mu^{(3)}(p, p', r) \equiv e \delta(p' - p - r) \Gamma_\mu^{(3)}(p, p') \quad (4.12)$$

in the same way as we deduced the mass operator, we get from the equation of motion (2.33), (2.20) and (2.24)

$$\Gamma_\mu^{(3)}(p, p', r) = \delta(p' - p - r) e \Gamma_\mu(p, p') + (2\pi)^4 \frac{\delta}{\delta A^\mu(-r)} (\Sigma(p, p') + \Sigma'(p, p')). \quad (4.13)$$

In this expression

$$\Sigma(p, p') = i \frac{e^2}{(2\pi)^4} \int dk dq D^{\alpha\beta}(k - p, q) \Gamma_\alpha(p, k) S(k, p' + q) \Gamma_\beta(p' + q, p') \quad (4.14)$$

is the mass operator at the one loop order, $\Gamma_\mu(p, p')$ is given by the definition (2.25) and

$$\begin{aligned} \Sigma'(p, p') = & \int d\bar{k} d\bar{k}' dp'' \Gamma_{db}^{(2)}(p'', p') \left\{ e^2 \left[\frac{1}{2} F_2(p - k, k - k') \rho_1^2(k) \not{k} \right. \right. \\ & + \int dq \rho_2(k) \rho_2(k - q) \rho_3(q) \Gamma_\nu(q) F_1(p - k, k - k' - q) \frac{\delta}{\delta J_\nu(-q)} \Big]_{ac} S_{cd}(k', p'') \\ & + e^3 \frac{i}{2} \left[\frac{1}{3} F_3(p - k, k - k') \rho_1^2(k) \not{k} + \int dq \rho_2(k) \rho_2(k - q) \rho_3(q) \Gamma_\nu(q) \right. \\ & \left. \left. F_2(p - k, k - k' - q) \frac{\delta}{\delta J_\nu(-q)} \right]_{ac} S_{cd}(k', p'') + \mathcal{O}(\Lambda^\Delta) \right\} \gamma^G. \end{aligned} \quad (4.15)$$

If we keep only the terms of order e^3 in the expression (4.15) and use the recursion formulas (B.5) we can show respectively that the first and the second bracket of Σ' vanish on the one hand after integration over dk and on the other hand after successive integration over dp'', dk' and dk . Then according to the relation (3.5), the functional derivative of Σ (4.14) with respect to the C-number function A^μ is easily obtained. We thus get for the vertex function Fig.5 at the one loop level

$$\Gamma_\mu^{(3)}(p, p') = \gamma_\mu + \frac{ie^2}{(2\pi)^4} \int dk D^{\alpha\beta}(k - p) \Gamma_\alpha(p, k) S(k) \Gamma_\mu(k, k + r) S(k + r) \Gamma_\beta(k + r, p'). \quad (4.16)$$

As the (UV) cutoff Λ tends to infinity due to the property (2.26), we recover formally the standard non regularized expression for the vertex function.

Now we work in Feynman gauge and assume that the external electron lines are on mass-shell, thus we use freely the Gordon relation and express as usual the vertex function as

$$\Gamma_\mu^{(3)}(p, p') = \gamma_\mu (1 + F_1(r^2)) + \frac{i}{2m} \sigma^{\mu\nu} r_\nu F_2(r^2), \quad (4.17)$$

where $r = p' - p$ is the momentum transfer. In this case the computation can be simplified if one notices that the term $\rho_1^2(p) \not{p} - \rho_1^2(k) \not{k}$ which occurs in $\Gamma_\mu(p, k)$ (2.25) is identical to $S^{-1}(p) - S^{-1}(k)$ and that

$$\bar{u}(p) S^{-1}(p) = \bar{u}(p) \mathcal{O}(\frac{\infty}{*\epsilon}), \quad (4.18)$$

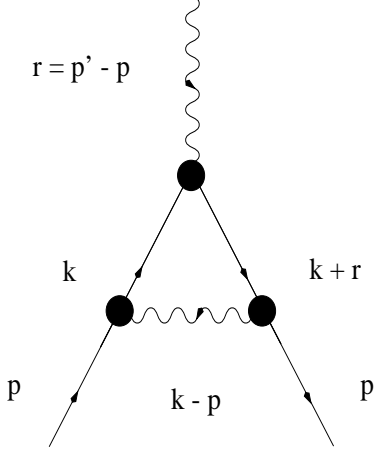


Figure 5: The vertex function.

p being the momentum of the outgoing electron. A similar reduction occurs for the term $\Gamma_\beta(k+r, p')u(p')$. We give to the photon an arbitrary small mass κ in order to avoid the (IR) divergence and substitute the free electron propagators by their expressions (3.15). After Feynman parametrization of the product of propagators, we perform the integration over the internal momentum using the formulas of appendix C. Then, if we define the integrals

$$\begin{aligned} I_1(r^2) &= \int_0^1 \frac{dx}{m^2 - r^2 x(1-x)} & I_3(r^2) &= \int_0^1 dx \log \frac{m^2 - r^2 x(1-x)}{\mu^2} \\ I_2(r^2) &= \int_0^1 \frac{dx}{m^2 - r^2 x(1-x)} \log \frac{m^2 - r^2 x(1-x)}{\mu^2} \end{aligned} \quad (4.19)$$

we get for the forms factors $F_1(r^2)$ and $F_2(r^2)$ the expressions

$$\begin{aligned} F_1(r^2) &= \frac{\alpha}{\pi} \left\{ \frac{1}{2} \log \frac{\Lambda}{\mu} - \frac{1}{4} \gamma - \frac{3}{8} - \frac{1}{2} \log 2 - \frac{1}{4} (2m^2 - r^2) I_2(r^2) + \frac{1}{2} \left[(2m^2 - r^2) \log \frac{\kappa}{\mu} \right. \right. \\ &\quad \left. \left. + (3m^2 - r^2) \right] I_1(r^2) - \frac{1}{4} I_3(r^2) \right\} \end{aligned} \quad (4.20)$$

$$F_2(r^2) = \frac{\alpha}{2\pi} m^2 I_1(r^2). \quad (4.21)$$

μ being an arbitrary unit of mass. As expected from (4.17) and (4.21) we recover the known result for the anomalous magnetic moment of the electron [17]. The expression of the form factor $F_1(r^2)$ which is (IR) and (UV) divergent is also standard.

What about the prediction of the (WT) identity? From the general relation (2.34) we can deduce the identity

$$r^\mu \Gamma_\mu^{(3)}(p, p', r) = e \rho_3(r) \left[\Gamma^{(2)}(p+r, p') - \Gamma^{(2)}(p, p'-r) \right], \quad r = p' - p. \quad (4.22)$$

If Z_1 and Z_2 are respectively the renormalization constants of the vertex function and of the external electron lines associated with the mass operator i.e.

$$Z_2 - 1 = - \left. \frac{\partial \Sigma(p)}{\partial \not{p}} \right|_{\not{p}=m} \quad (4.23)$$

the identity (4.22) implies that $Z_1 Z_2^{-1}$ must be finite. If the external electron lines are off mass-shell, we can show that a term proportional to $\alpha \not{r}_\mu / r^2$ contributes to the vertex function too. Since the derivative of this term is not regular near the origin, the expression $r^\mu \partial \Gamma_\mu^{(3)} / \partial r^\rho$ does not vanish as $r^\mu \rightarrow 0$. As a consequence, the (WT) (4.22) does not imply the equality of Z_1 and Z_2 . This explains the reason why Z_1 differs from Z_2 in first order in α by a numerical constant as it can be easily seen.

5 Conclusion

We have constructed in four dimensions a continuous regularization scheme of QED based on a non local extension of the action. Since the measure of the path integral is invariant under the gauge group, this regularization scheme is actually non-perturbative. Once the regularized action is fixed by the choice of the cutoff functions, the regularized amplitudes can be calculated in a straightforward way from the path integral without the need to adjust some parameters or to define some integrals formally. In order to illustrate this fact we have deduced the regularized (1PI) functions at the one loop order from the equations of motion, which are now mathematically well-defined objects and thus can be represented by regularized Feynman diagrams. It follows from the expression (2.12) of the C-number function $C_\mu(z, z', x)$, which was imposed by the necessity of gauge invariance, that the theory keeps trace of the non-locality in some potentially divergent (1PI) functions through additional terms which are regular in the (IR) domain, but whose derivatives are not. This is the case for the polarization operator and for the vertex function if this latter is calculated off mass-shell. Although the polarization operator is transverse, it contains a quadratically divergent term whose derivative is not regular in the (IR) domain. Contrary to known gauge invariant regularization procedures, like the Pauli -Villars regularization where the quadratic divergent piece is not gauge invariant and then legislated to zero, this term must be removed by a specific choice of the counterterm S_Λ . In the case of massless QED some terms of the non gauge invariant part of closed fermions loops are not unambiguously regularized to zero. This fact is reminiscent of the Nielsen-Ninomiya Theorem for chiral fermions on the lattice.

Despite the fact that we recover the known result for the anomalous magnetic moment of the electron at the one loop order, we must investigate all the implications of the (IR) behavior of this regularized form of QED for physical process. These questions, the problem of non-perturbative renormalization, the possibility of dynamical mass generation for the photon through a new non-local interaction and the extension of this regularization scheme to non-abelian gauge theories like QCD, where non-perturbative effects are known to occur, will be investigated elsewhere.

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Appendix A

The Γ_μ matrices satisfy :

$$\{\Gamma_\mu(r), \Gamma_\nu(r)\} = \{\Gamma_\mu(r), \gamma_\nu\} = 2(g_{\mu\nu} - \frac{r_\mu r_\nu}{r^2}) \quad (\text{A.1})$$

$$r_\mu \Gamma_\mu(r) = 0, \quad \gamma^\nu \Gamma_\mu(r) \gamma_\nu = -2\Gamma_\mu(r) \quad (\text{A.2})$$

Appendix B

The Fourier transform of $e^{ieL(x,y)}$ is

$$F(p, q) = \int dx dy e^{i(px+qy)} e^{ieL(x,y)}. \quad (\text{B.1})$$

If we define

$$F_n(p, q) = \int dx dy e^{i(px+qy)} L^n(x, y) \quad (\text{B.2})$$

we have the expansion

$$F(p, q) = \sum_{n=0}^{+\infty} \frac{(ie)^n}{n!} F_n(p, q) \quad (\text{B.3})$$

$$F_0(p, q) = ((2\pi)^4)^2 \delta(p) \delta(q), \quad (\text{B.4})$$

and the following recursion formulas hold for the F_n 's

$$F_{n+1}(p, q) = -i \int d\bar{r} \rho_3(r) \frac{r_\mu}{r^2} (F_n(p-r, q) - F_n(p, q-r)) A^\mu(r). \quad (\text{B.5})$$

In addition one can easily show by induction that

$$\int d\bar{k} k_\mu F_n(p-k, k-p') = 0, \quad \int d\bar{k} F_n(p-k, k-p') = 0, \quad (\text{B.6})$$

for all p and p' .

Appendix C

If we take as usual the functional derivative of the electron source $\eta_a(k)$ (2.31) with respect to $\eta_b(k')$, the inverse of the full electron propagator (3.3)

$$\Gamma_{ab}^{(2)}(k, k') \equiv -(2\pi)^4 \frac{\delta^2 \Gamma}{\delta \psi_a(k) \delta \psi_b(k')} \quad (\text{C.1})$$

is defined by

$$\int dk'' \Gamma_{ac}^{(2)}(k, k'') S_{cb}(k'', k') = \delta_{ab} \delta(k - k'). \quad (\text{C.2})$$

In the same manner the inverse of the full photon propagator (4.3)

$$\Gamma_{\mu\nu}^{(2)}(k, k') \equiv (2\pi)^4 \frac{\delta^2 \Gamma}{\delta A_\mu(k) \delta A_\nu(-k')} \quad (\text{C.3})$$

verifies

$$\int dk'' \Gamma_{\mu\alpha}^{(2)}(k, k'') D_{\alpha\nu}(k'', k') = \delta_{\mu\nu} \delta(k - k'). \quad (\text{C.4})$$

The relations (C.2) and (C.4) are the basic tools for expressing, by means of Schwinger's technique of functional differentiation, general Green's functions in terms of (1PI) functions and propagators. For instance, taking the functional derivative of (C.2) with respect to the C-number function $A^\mu(-r')$, we obtain the relation

$$\frac{\delta S_{ba}(p, k)}{\delta A^\mu(-r')} = - \int dk' dk'' S_{bc}(p, k'') \frac{\delta \Gamma_{cd}^{(2)}(k'', k')}{\delta A^\mu(-r')} S_{da}(k', k). \quad (\text{C.5})$$

Appendix D

In this appendix we give the asymptotic form of the integrals which we encounter in the calculation of the relevant (1PI) functions. We start with the evaluation of the integral

$$I = \int dk \frac{e^{\epsilon(ak^2 + 2bk)}}{(-k^2 - 2qk + C)^\alpha}. \quad (\text{D.1})$$

where ϵ is an infinitesimal parameter. At first we make the shift of variable $k \rightarrow k - q$ and we express the denominator of the integrand of (D.1) by means of Schwinger's parametric integral. After a Wick rotation the integral over dk is easily performed [4], and we get

$$I = i \frac{\pi^2}{\Gamma(\alpha)} e^{\epsilon(aq^2 - 2bq)} \int_0^{+\infty} du u^{\alpha-1} e^{-u(q^2 + C)} \frac{1}{(a\epsilon + u)^2} e^{-\epsilon^2 \frac{(b-aq)^2}{a\epsilon + u}} \quad (\text{D.2})$$

Now if we expand $\exp[-\epsilon^2(b-aq)^2/(a\epsilon + u)]$ in power series and integrate over du we obtain the integral I in terms of the degenerate hypergeometric function $\Psi(\alpha, n, z)$ [18] as

$$I = i\pi^2 e^{\epsilon(aq^2 - 2bq)} (a\epsilon)^{\alpha-2} \left\{ \Psi\left(\alpha, \alpha-1, (q^2 + C)a\epsilon\right) + \sum_{\nu=1}^{+\infty} \frac{(-1)^\nu}{\nu!} \epsilon^\nu (b-aq)^{2\nu} a^{-\nu} \right. \\ \left. \Psi\left(\alpha, \alpha-1-\nu, (q^2 + C)a\epsilon\right) \right\}. \quad (\text{D.3})$$

When n is an integer, the degenerate hypergeometric function $\Psi(\alpha, n, z)$ can be expanded as [19]

$$\Psi(\alpha, n, z) = \frac{(-1)^{n+1}}{\Gamma(\alpha-n+1)} \left\{ \sum_{k=1}^{n-1} \frac{(-1)^k (k-1)!}{(n-k-1)!} \frac{\Gamma(\alpha-k)}{\Gamma(\alpha)} z^{-k} + \sum_{k=0}^{+\infty} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \frac{z^k}{(n+k-1)!k!} \right. \\ \left. [\psi(k+1) + \psi(n+k) - \psi(\alpha+k) - \log z] \right\}, \quad n \geq 1 \quad (\text{D.4})$$

with the convention that for $n = 1$ the first sum is identically zero. In addition the relation [19]

$$\Psi(\alpha, n, z) = z^{1-n} \Psi(\alpha - n + 1, 2 - n, z) \quad (\text{D.5})$$

holds. In the above formulas, $\psi(x)$ is the ψ function [18]. For $\alpha = 2$, we use (D.4) and (D.5) and the integral (D.1) becomes

$$\int dk \frac{e^{\epsilon(ak^2+2bk)}}{(-k^2-2qk+C)^2} = i\pi^2 \left(-\log a\epsilon + \psi(1) - 1 - \log(q^2 + C) + \mathcal{O}(\epsilon) \log \epsilon \right). \quad (\text{D.6})$$

Similarly for the following integrals we obtain the final result

$$\begin{aligned} \int dk \frac{k_\mu e^{\epsilon(ak^2+2bk)}}{(-k^2-2qk+C)^2} &= i\pi^2 \left[q_\mu \log a\epsilon - \frac{1}{2} b_\mu a^{-1} - q_\mu \left(\psi(1) - \frac{3}{2} - \log(q^2 + C) \right) \right. \\ &\quad \left. + \mathcal{O}(\epsilon) \log \epsilon \right] \end{aligned} \quad (\text{D.7})$$

$$\begin{aligned} \int dk \frac{k_\mu k_\nu e^{\epsilon(ak^2+2bk)}}{(-k^2-2qk+C)^2} &= i\pi^2 \left[-\frac{1}{4} g_{\mu\nu} (a\epsilon)^{-1} - \left(\frac{1}{2} g_{\mu\nu} (q^2 + C) + q_\mu q_\nu \right) \log a\epsilon \right. \\ &\quad + \frac{1}{6} (b - aq)_\mu (b - aq)_\nu a^{-2} + \frac{1}{2} g_{\mu\nu} (q^2 + C) \left(\psi(1) - \frac{1}{2} - \log(q^2 + C) \right) \\ &\quad + \frac{1}{12} g_{\mu\nu} (b - aq)^2 a^{-2} + \frac{1}{2} \left((b - aq)_\mu q_\nu + (b - aq)_\nu q_\mu \right) a^{-1} \\ &\quad \left. + q_\mu q_\nu \left(\psi(1) - 1 - \log(q^2 + C) \right) + \mathcal{O}(\epsilon) \log \epsilon \right] \end{aligned} \quad (\text{D.8})$$

$$\begin{aligned} \int dk \frac{k_\mu k_\nu e^{\epsilon(ak^2+2bk)}}{(-k^2-2qk+C)^3} &= i\frac{\pi^2}{2} \left[\frac{1}{2} g_{\mu\nu} \left(\log a\epsilon - \psi(1) + \frac{3}{2} + \log(q^2 + C) \right) \right. \\ &\quad \left. + \frac{q_\mu q_\nu}{q^2 + C} + \mathcal{O}(\epsilon) \log \epsilon \right] \end{aligned} \quad (\text{D.9})$$

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